GENERAL AND SET THEORETIC TOPOLOGY SYLLABUS

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Prerequisite

- **set theory**: operation on sets, cardinals, ordinals, cardinal and ordinal arithmetic, cofinalities, König lemma, transfinite induction, transfinite recursion, Zorn lemma.
- **topology**: the notion of topological spaces, bases, metric spaces, subspaces, continuous images, Cartesian products,

Literature.

- Willard, Stephen; *General topology*, Addison-Wesley, 1970
- Engelking, Ryszard *General topology,* Sigma series in pure mathematics ; 6. Heldermann Verlag, 1989.
- Juhász,I; *Cardinal functions in topology - ten years later* (Mathematical centre tracts ; 123, 1980.
- *Open Problems In Topology II*, Elliott Pearl, Toronto, Canada
- Handbook of Set-theoretic Topology

1. General topology

1.1. Axioms of separation.

- T_0, T_1, and T_2 (or Hausdorff) spaces
- T_3 (or regular) spaces
- $T_{3.5}$ (or completely regular) spaces
- T_4 (or normal) spaces

Non-trivial examples:

- a Hausdorff space which is not regular.[E, 1.5.6]
- a regular, not completely regular space [E, 1.5.9].
- a regular, not completely Hausdorff space, [W, 18G]
- Sorgenfrei line [E, 1.2.2].
- Niemytzki plane, [E, 1.2.4], [E, 1.5.10].

— (Urysohn's Lemma).] For every pair A,B of disjoint closed subsets of a normal space X there exists a continuous function $f : X \to I$ such that $f(x) = 0$ for $x \in A$ and $f(x) = 1$ for $x \in B$. [E, 1.5.11.].

— Every regular space of countable weight is normal. [E, 1.5.16.].

A topological space X is called a *hereditarily normal space, (T_5)* if every subspace of X is a normal space.

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A topological space X is called a *perfectly normal space, (T_6)* if X is a normal space and every closed subset of X is a G_δ-set.
Results:

- — A subset S of a normal space X is a closed G_δ-set if and only if there exists a continuous function $f : X \to I$ such that $A = f^{-1}(0)$. [E, 1.5.12.].
- — For every point-finite open cover $\{U_s : s \in S\}$ of a normal space X there exists an open cover $\{V_s : s \in S\}$ of X such that $\overline{V_s} \subset U_s$ for every $s \in S$. [E, 1.5.18.]
- — The Vedenissoff Theorem. [E, 1.5.19.]
- Let $F = \langle 2, 3 \rangle$.
- The Alexandroff cube of weight $m \geq \omega$ is F^m.
- The Tychonoff cube of weight $m \geq \omega$ is the space I^m.
- — The Alexandroff cube F^m is universal for all T_0-spaces of weight $m \geq \omega$. [E, 2.3.26.]
- — The Tychonoff cube is universal for all Tychonoff spaces of weight $m \geq \omega$. [E, 2.3.23]

1.2. **Basic cardinal functions.**
Definitions:

- the weight of a space, $w(X)$
- the character of a space, $\chi(X)$
- the density of a space, $d(X)$

Results:

- If $w(X) \leq m$, then for every family $\mathcal{U}$ of open subsets of X there exists a set $\mathcal{U}'$ such that $|\mathcal{U}'| \leq m$ and $\bigcup \mathcal{U}' = \bigcup \mathcal{U}$. [E, 1.1.14.]
- If $w(X) \leq m$ then for every base $\mathcal{B}$ for X there exists a base $\mathcal{B}' \subset \mathcal{B}$ such that $|\mathcal{B}'| \leq m$. [E, 1.1.15.]

Inequalities:

- For every T_0-space we have $|X| \leq 2^{w(X)}$.[E, 1.5.1.]
- For every Hausdorff space X we have $|X| \leq 2^{2^{d(X)}}$ and $|X| \leq d(X)^{\chi(X)}$.[E, 1.5.3.]
- For every regular space X we have $w(X) \leq 2^{d(X)}$. [E, 1.5.7.]

1.3. **Operation on topological spaces.**

- — Any subspace of a T_i-space is a T_i-space for $i \leq 3.5$. Normality is hereditary with respect to closed subsets. Perfect normality is a hereditary property. [E, 2.1.6.]
- — (The Tietze-Urysohn Theorem). Every continuous function from a closed subspace M of a normal space X to I or R is continuously extendable over X. [E, 2.1.8]
- — No separable normal space contains a closed discrete subspace of cardinality continuum. [E, 2.1.10.]
- — Any Cartesian product of T_i-spaces is a T_i-space for $i \leq 3.5$. [E, 2.3.11.]
- — The Sorgenfrey line K is perfectly normal, so that K is hereditarily normal as well. It turns out that the Cartesian product K x K is not normal. [E, 2.3.12.]
- — (The Hewitt-Marczewski-Pondiczery Theorem.) If $d(X_S) \leq m$ for every $s \in S$ and $|S| \leq 2^m$, then $d(\prod_{s \in S} X_s) \leq m$. [E, 2.3.15.]

— A countable space with character $> \omega$ at all points. [E, 2.3.37.]

- Quotient space,
- limit of inverse systems

1.4. **Compact spaces.**

— A Hausdorff space X is compact if and only if every family of closed subsets of X which has the finite intersection property has non-empty intersection. [E, 3.1.1.]
— Every compact space is normal. [E, 3.1.9.]
— For every compact space X we have nw(X) = w(X). [E, 3.1.19.]
— For every compact space X we have $w(X) \leq |X|$. [E, 3.1.21.]
— If a compact space Y is a continuous image of a topological space X, then $w(Y) \leq w(X)$. [E, 3.1.22.]
— Arhangelskii. For every infinite compact space X we have $|X| \leq 2^{\chi(X)}$. [E, 3.1.29.]
— (Alexander Subbase Theorem [1939]). Let X be a Hausdorff space and P a subbase for X; show that the space X is compact if and only if every cover of X by members of P has a finite subcover (this is the Alexander subbase theorem). [E, 3.12.2]
— The Tychonoff Theorem. The Cartesian product of compact spaces is compact. [E, 3.2.4.]
— The Stone-Weierstrass Theorem. If a ring P of continuous real-valued functions defined on a compact space X contains all constant functions, separates points and is closed with respect to uniform convergence (i.e., is a closed subset of the space Rx with the topology of uniform convergence), then P coincides with the ring of all continuous real-valued functions defined on X. [E, 3.2.21.]

- locally compact spaces
- the Alexandroff double circle. [E, 3.1.26.]
- $\omega_1 + 1$ [E, 3.1.27.]

- A pair (Y, c), where Y is a compact space and $c : X \to Y$ is a homeomorphic embedding of X in Y such that $\overline{c(X)} = Y$, is called a *compactification* of the space X.

— A topological space X has a compactification if and only if X is a Tychonoff space. [E, 3.5.1.]
— Every Tychonoff space X has a compactification (Y,c) such that w(Y) = w(X). [E, 3.5.2.]
— For every compactification Y of a space X we have $|Y| \leq 2^{2^{d(X)}}$ and $w(Y) \leq 2^{d(X)}$ [E, 3.5.3.]

- We shall now define an order relation in the family C(X). Let $c_2X \leq c_1X$ if there exists a continuous mapping $f : c_1X \to c_2X$ such that $fc_1 = c_2$;

— For every Tychonoff space X there exists in C(X) a largest element with respect to the order $\leq$. [E, 3.5.10.]

- The largest element in C(X) is called the Cech-Stone compactification of X or the maximal compactification of X and is denoted by βX;

— The Alexandroff Compactification Theorem. Every non-compact locally compact space X has a compactification uX with one-point remainder. This compactification is the smallest element in C(X) with respect to the order ¡, its weight is equal to the weight of the space X. [E, 3.5.11.]

— If a compact space Y is a continuous image of the remainder $cX \setminus c(X)$ of a compactification cX of a locally compact space X, then the space X has a compactification $c'X \leq cX$ with the remainder homeomorphic to the space Y. [E, 3.5.13.]

— Every continuous mapping $f : X \to Z$ of a Tychonoff space X to a compact space Z is extendable to a continuous mapping $F : \beta X \to Z$. If every continuous mapping of a Tychonoff space X to a compact space is continuously extendable over a compactification aX of X, then aX is equivalent to the Cech-Stone compactification of X. [E, 3.6.1.]

— For every $m \geq \omega$ the Cech-Stone compactification of the space D(m) has cardinality 2^{2^m} and weight 2^m. [E, 3.6.11.]

— Every infinite closed set $F \subset \beta N$ contains a subset homeomorphic to βN; in particular F has cardinality 2^{2^ω} and weight 2^ω. βN does not contain convergent sequences. [E, 3.6.14.]

- We say that a topological space X is a Lindelof space, or has the Lindelof property, if X is regular and every open cover of X has a countable subcover.

— Every Lindelof space is normal.[E, 3.8.2]

- A topological space X is called a countably compact space if X is a Hausdorff space and every countable open cover of X has a finite subcover.

— A topological space is compact if and only if it is a countably compact space with the Lindelof property. [E, 3.10.1.]

— There are two countably compact Tychonoff spaces X and Y such that the Cartesian product X x Y is not countably compact. They are subspaces of βN satisfying the conditions $X \cup Y = \beta N$ and $X \cap Y = N$. [E, 3.10.19.]

- A topological space X is called pseudocompact if X is a Tychonoff space and every continuous real-valued function defined on X is bounded.
- A topological space X is called sequentially compact if X is a Hausdorff space and every sequence of points of X has a convergent subsequence.

— The Cartesian product of a countably compact space X and a sequentially compact space Y is countably compact. [E, 3.10.36.]

1.5. **Metric spaces.**

— Every compact metrizable space is separable. [E, 4.1.18.]

— The Hilbert cube I^ω is universal for all compact metrizable spaces and for all separable metrizable spaces. [E, 4.2.10]

- A family $\{A_t\}_{t \in T}$ of subsets of a topological space X is *locally finite* if for every point $x \in X$ there exists a neighbourhood U such that the set $\{t \in T : Un \cap A_t \neq \emptyset\}$ is finite. If every point $x \in X$ has a neighbourhood that intersects at most one set of a given family, than we say that the family is *discrete*.
- A family of subsets of a topological space is called σ-locally finite (σ-discrete) if it can be represented as a countable union of locally finite (discrete) families.

— The Stone Theorem. Every open cover of a metrizable space has an open refinement which is both locally finite and σ-discrete. [E, 4.4.1.]

— Every metrizable space has a σ-discrete base.[E, 4.4.3.]
— The Nagata-Smirnov Metrization Theorem. A topological space is metrizable if and only if it is regular and has a σ-locally finite base.[E, 4.4.7.]
— The Bing Metrization Theorem. A topological space is metrizable if and only if it is regular and has a σ-discrete base.[E, 4.4.8.]
— The Cartesian product $[J(m)]^\omega$ of ω copies of the hedgehog $J(m)$ is universal for all metrizable spaces of weight $m \geq \omega$ [E, 4.4.9.]

1.6. **Paracompactness.**

- A topological space X is called a paracompact space if X is a Hausdorff space and every open cover of X has a locally finite open refinement.
- A topological space X is called *collectionwise normal* if X is a T_1-space and for every discrete family $\{F_s\}_{s\in S}$ of closed subsets of X there exists a discrete family $\{V_s\}_{s\in S}$ of open subsets of X such that $F_s \subset V_s$ for every $s \in S$.

— Every compact space is paracompact.[E, 5.1.1.]
— Every Lindelof space is paracompact.[E, 5.1.2.]
— Every metrizable space is paracompact.[E, 5.1.3.]
— Every paracompact space is normal. [E, 5.1.5.]
— Every paracompact space is collectionwise normal.[E, 5.1.18.]
— ω_1 is not paracompact. Since it is countably compact and normal, it is collectionwise normal.[E, 5.1.21.]
— The Michael Theorem. Paracompactness is an invariant of closed mappings. [E, 5.1.33.]
— Let M be a subspace of a topological space X. One easily checks that the family of all sets of the form $U \cup K$, where U is an open subset of X and $K \subset X \setminus M$, is a topology on X; the set X with this new topology will be denoted by XM. [E, 5.1.22.]
— Michael'S Example. Denote by Q and P the subspaces of R consisting of all rational and all irrational numbers respectively. By virtue of Example 5.1.22 the space $X = R_Q$ is hereditarily paracompact (the space X is called the Michael line). We shall prove that the Cartesian product $X \times Y$, where Y = P, is not normal.[E, 5.1.32.]

- countably paracompact spaces

— A topological space X is normal and countably paracompact if and only if the Cartesian product $X \times I$ of X and the closed unit interval I is normal. [E, 5.2.8.]

- A Dowker space is a normal space X such that $X \times I$ is not normal.
- A sequence $W_1, W_2, ...$ of covers of a topological space X is called a development for the space X if all covers W_i are open, and for every point $x£ \in X$ and any neighbourhood U of x there exists a natural number i such that $St(x, WI) \subset U$.
- A sequence $W_1, W_2, ...$ of covers of a topological space X is called a strong development for the space X if all covers W_i are open, and for every point $x£ \in X$ and any neighbourhood U of x there exists a natural number i and a neighbourhood V of X such that $St(V, WI) \subset U$.

— Bing'S Metrization Criterion. A topological space is metrizable if and only if it is collectionwise normal and has a development.[E, 5.4.1.]
— The Moore Metrization Theorem. A topological space is metrizable if and only if it is a T_0-space and has a strong development.[E, 5.4.2.]

1.7. **Connected spaces.**

- We say that a topological space X is connected if X cannot be represented in the form $X1 \oplus X2$, where $X1$ and $X2$ are non-empty subspaces of X.
- The quasi-component of a point x in a topological space X is the intersection of all closed- and-open subsets of X which contain the point x.
- The component of a point x in a topological space X is the union of all connected subspaces of X which contain the point x.
- A topological space X is called a continuum if X is both connected and compact.

— The intersection $\bigcap_{i\in\omega} C_i$ of a decreasing sequence $C_1 \supset C_2 \supset \ldots$ of continua is a continuum. [E, 6.1.19.]
— The component C of a point x in a topological space X is contained in the quasi-component Q of the point x. [E, 6.1.22.]
— In a compact space X the component of a point x S X coincides with the quasi-component of the point x. [E, 6.1.23.]
— There is a subspace of $\mathbb{R}^2$ in which components and quasi-components are different from each other. [E, 6.1.24. .]
— The Sierpinski Theorem. If a continuum X has a countable cover by pair- wise disjoint closed subsets $\{X_i\}_{i\in\omega}$, then at most one of the sets X_i is non-empty. [E, 6.1.27.]

- A topological space X is called hereditarily disconnected if X does not contain any connected subsets of cardinality larger than one.
- A topological space X is called zero-dimensional if X is a non-empty T_1-space and has a base consisting of open-and-closed sets.
- A topological space X is called extremally disconnected if X is a Hausdorff space and for every open set $U \subset X$ the closure U is open in X.

— Every zero-dimensional space is hereditarily disconnected. [E, 6.2.1]
— Erdős's Example. a hereditarily disconnected separable metric space which is not zero-dimensional . Let X be the subspace of Hilbert space 77, defined in [E, Example 4.1.7] consisting of all infinite sequence of rational numbers. The space X is hereditarily disconnected. [E, 6.2.19.]
— The space βN is extremally disconnected. [E, 6.2.29]

2. Set Theoretic Topology

2.1. **Cardinal functions.**

- width $d(X)$, character $\chi(X)$, density $d(X)$, net-weight $nw(X)$, pseudo-character $\psi(X)$, tightness $t(X)$, Lindelöf degree $L(X)$, spread $s(X)$, extent $e(X)$, cellularity $c(X)$, hereditary Lindelöf degree $h(X)$, hereditary density $z(X)$, number of open sets, number or regular open sets,

Basic results: Assume that X is T_2.

— deGroot: $|X| \leq 2^{h(X)}$.
— Hajnal-Juhasz, $|X| \leq 2^{\chi(X)c(X)}$.
— Hajnal-Juhasz, $|X| \leq 2^{\psi(X)s(X)}$.
— Hajnal-Juhasz, $|X| \leq 2^{2^{s(X)}}$.
— Arhangelski, $|X| \leq 2^{\chi(X)L(X)}$.
— Shapirowski, $|X| \leq 2^{t(X)\psi(X)L(X)}$
— Hajnal-Juhasz, $z(X) \leq 2^{s(X)}$

Inequalities on special spaces

— Bell, Ginsburg, Woods.If X is T_4, then $|X| \leq 2^{\chi(X)wL(X)}$

Assume that X is compact T_2.

— $\psi(X) = \chi(X)$.
— $psw(X) = nw(X) = w(X)$
— Arhangelski, $t(X) = F(X)$
— Sapirovski, $z(X) \leq s(X)^+$
— Cech-Pospisil, if $\chi(x, X) \geq \kappa$ for each $x \in X$, then $|X| \geq 2^\kappa$.

- $\chi(X) = \omega$ then $|X| \leq \omega$ or $|X| = 2^\omega$

Examples

- Ostasewsky's construction
- HFDs and HFCs

2.2. **Combinatorial principles.**

2.2.1. *CH, $\diamondsuit$ and $\clubsuit$.* E.g.: If CH holds, then there is a separable, first countable, countably compact, noncompact space.

2.2.2. *Martin's Axioms.*

- countable chain condition, ccc
- filter, generic filter,
- $MA(\kappa)$

Topological reformulation: No compact Hausdorff space with the ccc can be the union of less than 2^ω nowhere dense subsets.

— If $MA(\omega_1)$ holds, then there is no Suslin line, i.e. every c.c.c ordered space is separable.
— If CH holds, then There are two ccc partial orders P_1 and P_2 such that $P_1 \times P_2$ is not ccc.
— If $MA(\kappa)$ holds, then every partial order of cardinality κ with the ccc is also σ-centered.

MA for restricted kinds of partial orders

2.2.3. *PFA and its consequences.*

- proper poset, PFA

Assume that PFA holds

- PFA implies that every T_2-space of countable spread has cardinality $\leq 2^\omega$.
- Balogh: PFA implies that each compact space of countable tightness is sequential.

2.3. **Cardinal invariants of the reals.** Cichon's diagramm

2.4. **Selected problems.**

2.4.1. *S and L spaces.*

- An L-space is a regular, hereditarily Lindelöf, but not separable space.
- An S-space is a regular, hereditarily separable, but not Lindelöf space.

— The existence of an S-space is independent
— There is an L-space

Main problem: Is there an L-group? Is there a space X such that X^2 is an L-space?

2.4.2. *Jakovlev spaces.* Main problem: Is there a Jakovlev space?

2.4.3. *Dowker spaces.* A normal space whose product with the closed unit interval I is not normal is called a Dowker space.

Main problem: Is there s DOwker space of size ω_1?

2.4.4. *Splendid spaces.* . A countably compact and locally countable T_3 space is called *good*. A good space is *splendid* if countable subsets have countable (or equivalently, compact) closures.

ω_1 is splendid.

Main problem: Is there a good space of size continuum? Are there such spaces of arbitrarily large cardinality?

2.4.5. *Lindelöf G_δ spaces.* Assume that X is a regular space, $\psi(X) = L(X) = \omega$. Find lower and upper bounds of $|X|$.

2.4.6. *Nonmetrizable manifolds.*

2.5. **Questions.** Is a normal, linearly Lindelöf space Lindelöf ? Is a regular D-space Lindelöf ?

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